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Existence of Kirillov–Reshetikhin Crystals for Multiplicity-Free Nodes

by

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Abstract

We show that the Kirillov–Reshetikhin crystal $B^{r,s}$ exists when r is a node such that the Kirillov–Reshetikhin module $W^{r,s}$ has a multiplicity-free classical decomposition.

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§1. Introduction

Kirillov–Reshetikhin (KR) modules are a class of finite-dimensional representations of an affine quantum group $U_q'(\mathfrak{g})$ without the degree operator that is classified by their Drinfel'd polynomials, that have received significant attention. We denote a KR module by $W^{r,s}$, where r is a node of the classical (i.e. underlying finite type) Dynkin diagram and $s \in \mathbb{Z}_{>0}$. One construction of a KR module $W^{r,s}$ is by computing the minimal affinization of the highest weight $U_q(\mathfrak{g}_0)$-module $V(s\overline{\Lambda}_r)$ [Cha95, CP95a, CP96a, CP96b], where $\mathfrak{g}_0$ is the classical Lie algebra. Another method is by using the fusion construction of [KKM+92] from the image under an R-matrix of an s-fold tensor product of the fundamental module $W^{r,1}$ (see, e.g., [Kas02]). KR modules are also known to have special properties. The classical decomposition, the branching rule of $W^{r,s}$ to a $U_q(\mathfrak{g}_0)$-module, is given by a fermionic formula [DFK08, Her10], which leads to the (virtual) Kleber algorithm [Kle98, OSS03]. The characters (resp. q-characters) of KR modules also satisfy the Q-system (resp. T-system) relations [Her10, Nak03]. Furthermore, the graded characters of (Demazure submodules of) a tensor product of fundamental

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modules are (nonsymmetric) Macdonald polynomials at $t = 0$ [LNS+15, LNS+16a] ([LNS+17]).

One important (conjectural) property [HKO+99, HKO+02] is that the KR module $W^{r,s}$ admits a crystal base [Kas90, Kas91], which is known as a Kirillov–Reshetikhin (KR) crystal and denoted by $B^{r,s}$. Kashiwara showed that all fundamental modules $W^{r,1}$ have crystal bases [Kas02]. It was shown that $B^{r,s}$ exists in all nonexceptional types in [Oka07, OS08] and in types $G_2^{(1)}$ and $D_4^{(3)}$ in [KMOY07, Nao18, Yam98]. For all affine types, the existence of $B^{r,s}$ has been proven when r is adjacent to 0 or in the orbit of 0 under a Dynkin diagram automorphism (equivalently, $W^{r,s}$ is irreducible as $U_q(\mathfrak{g})$-module) [KKM+92].

Our main result is that the KR module $W^{r,s}$ has a crystal base whenever its classical decomposition is multiplicity-free in all affine types. We do this by showing the existence of $B^{r,s}$ in the cases not covered by [KKM+92, Oka07, OS08]. More explicitly, we show this for $r = 3, 5$ in type $E_6^{(1)}$, for $r = 2, 6$ in type $E_7^{(1)}$, for $r = 7$ in type $E_8^{(1)}$ and for $r = 4$ in types $F_4^{(1)}$ and $E_6^{(2)}$, where we label the Dynkin diagrams following [Bou02] (see also Figure 1 for the labeling). Using the techniques developed in [KKM+92], our proof shows the existence of a crystal pseudobase (L, B) by using the fusion construction of $W^{r,s}$ and is similar to [Oka07, OS08] by calculating the prepolarization for certain vectors. From there, we can construct the associated crystal by $B/\{\pm 1\}$.

Let us describe some possible applications of our results. The $X = M$ conjecture [HKO+99, HKO+02] arises from mathematical physics relating vertex models and the Bethe ansatz of Heisenberg spin chains, and the X side requires the existence of KR crystals. A uniform model for $B^{r,1}$ was given using quantum and projected level-zero LS paths [LNS+15, LNS+16b, LNS+16a, NS06, NS08a, NS08b]. Since the KR crystal $B^{r,s}$ exists, we have a partial (conjectural) combinatorial description from [LS19] using $(B^{r,1})^{\otimes s}$, partially mimicking the fusion construction.

After completion of this paper, we learned that Naoi independently proved all cases in type $E_6^{(1)}$ [NS19], which has since become a collaboration with the second author.

This paper is organized as follows. In Section 2 we give the necessary background. In Section 3 we show our main result: that the KR modules $W^{r,s}$ has a crystal pseudobase whenever $W^{r,s}$ has a multiplicity-free classical decomposition.

§2. Background

In this section we provide the necessary background.

Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra with index set I, Cartan matrix $A = (A_{ij})_{i,j \in I}$, simple roots $(\alpha_i)_{i \in I}$, simple coroots $(h_i)_{i \in I}$, fundamental weights

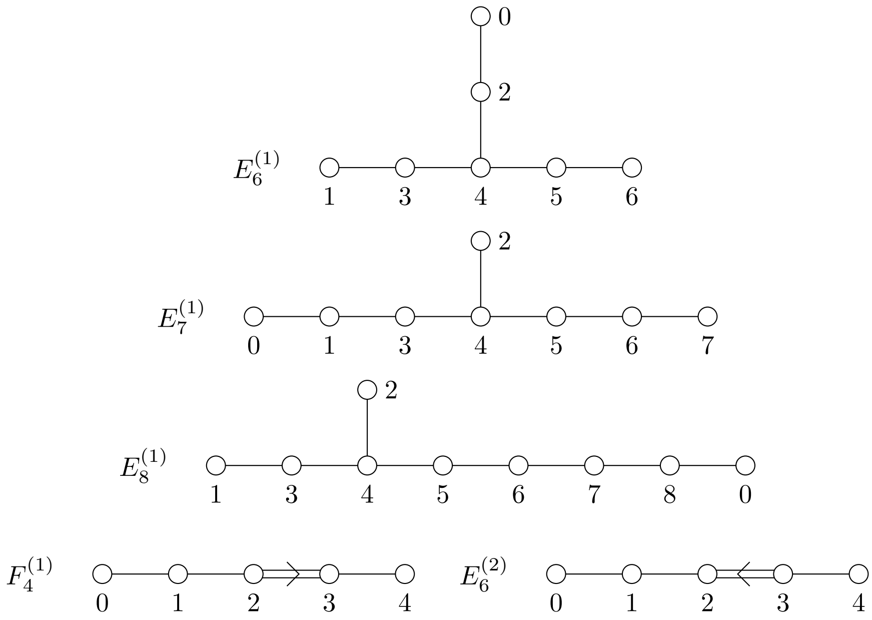

Figure 1. Dynkin diagrams for affine type $E_{6,7,8}^{(1)}$, $F_4^{(1)}$ and $E_6^{(2)}$.

$(\Lambda_i)_{i\in I}$, weight lattice P, dominant weights P^+, coweight lattice $P^\vee$ and canonical pairing $\langle\, ,\, \rangle\colon P^\vee \times P \to \mathbb{Z}$ given by $\langle h_i, \alpha_j\rangle = A_{ij}$. We note that we follow the labeling given in [Bou02] (see Figure 1 for the exceptional types and their labelings). Let $\mathfrak{g}_0$ denote the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$. Let $\overline{\lambda}$ denote the natural projection of $\lambda \in P$ onto the weight lattice P_0 of $\mathfrak{g}_0$, so $\{\overline{\Lambda}_r\}_{r\in I_0}$ are the fundamental weights of $\mathfrak{g}_0$.Let $\varpi_r = \Lambda_r - \langle c, \Lambda_r\rangle\, \Lambda_0$, where c is the canonical central element of $\mathfrak{g}$, denote the level-zero fundamental weights. Let q be an indeterminate, and we denote

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [k]_q! = [k]_q[k-1]_q \cdots [1]_q,$$
$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q[m-1]_q \cdots [m-k+1]_q}{[k]_q!},$$

for $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. Let $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$, where $(s_1, \ldots, s_n)$ is the diagonal symmetrizing matrix of A.

§2.1. Quantum groups

Let $U_q'(\mathfrak{g}) = U_q([\mathfrak{g}, \mathfrak{g}])$ denote the quantum group of the derived subalgebra of $\mathfrak{g}$. More specifically, the quantum group $U_q'(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra generated by e_i, f_i, q^h, where $i \in I$ and $h \in P^\vee$, that satisfies the relations

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee,$$
$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i \quad \text{for } h \in P^\vee, i \in I,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \text{for } i, j \in I,$$

and the *(quantum) Serre relations*

$$\sum_{k=0}^{1-A_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-A_{ij}-k)} = 0, \quad \sum_{k=0}^{1-A_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-A_{ij}-k)} = 0,$$

where $e_i^{(k)} = e_i^k / [k]_{q_i}!$ and $f_i^{(k)} = f_i^k / [k]_{q_i}!$ for all $i, j \in I$ such that $i \neq j$. We recall that $U_q'(\mathfrak{g})$ is a Hopf algebra; in particular, there exists a coproduct so we can take tensor products of $U_q'(\mathfrak{g})$-modules.

Denote the weight lattice of $U_q'(\mathfrak{g})$ by $P' = P/\mathbb{Z}\delta$, where δ is the null root of $\mathfrak{g}$. Therefore, there is a linear dependence relation on the simple roots in P'. As we will not be considering $U_q(\mathfrak{g})$-modules in this paper, we will abuse notation and denote the $U_q'(\mathfrak{g})$-weight lattice by P. For a $U_q'(\mathfrak{g})$-module M and $\lambda \in P$, we denote the λ weight space by

$$M_\lambda = \{v \in M \mid q^h v = q^{\langle h, \lambda \rangle} v \text{ for all } h \in P^\vee\}.$$

If $v \in M_\lambda \setminus \{0\}$, then we say $\mathrm{wt}(v) = \lambda$.

For $\lambda \in P_0^+$, we denote the highest weight $U_q(\mathfrak{g}_0)$-module by $V(\lambda)$.

§2.2. Crystal (pseudo)bases and polarizations

Let $\mathcal{A}$ denote the subring of $\mathbb{Q}(q)$ of rational functions without poles at 0. A *crystal base* of an integrable $U_q'(\mathfrak{g})$-module M is a pair (L, B), where L is a free $\mathcal{A}$-module and B is a basis of the $\mathbb{Q}$-vector space L/qL, such that

(1) $M \cong \mathbb{Q}(q) \otimes_{\mathcal{A}} L$,
(2) $L \cong \bigoplus_{\lambda \in P} L_\lambda$ with $L_\lambda = L \cap M_\lambda$,
(3) $\widetilde{e}_i L \subseteq L$ and $\widetilde{f}_i L \subseteq L$ for all $i \in I$,
(4) $B = \bigsqcup_{\lambda \in P} B_\lambda$ with $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,
(5) $\widetilde{e}_i B \subseteq B \sqcup \{0\}$ and $\widetilde{f}_i B \subseteq B \sqcup \{0\}$,
(6) $\widetilde{f}_i b = b'$ if and only if $\widetilde{e}_i b' = b$ for all $b, b' \in B$ and $i \in I$.

We say that (L, B) is a *crystal pseudobase* of M if it satisfies the conditions above for $B = B' \sqcup (-B')$, where B' is a basis of L/qL.

Let M be a $U_q'(\mathfrak{g})$-module. A *prepolarization* is a symmetric bilinear form $(\ ,\ ) \colon M \times M \to \mathbb{Q}(q)$ that satisfies

$$(2.1)\quad (q^h v, w) = (v, q^h w), \quad (e_i v, w) = (v, q_i^{-1} K_i^{-1} f_i w), \quad (f_i v, w) = (v, q_i^{-1} K_i e_i w)$$

for all $i \in I$, and $v, w \in M$.[1] Denote $\|v\|^2 = (v, v)$. If a prepolarization is positive definite with respect to the total order on $\mathbb{Q}(q)$,

$$f > g \text{ if and only if } f - g \in \bigsqcup_{n \in \mathbb{Z}} \{q^n(d + q\mathcal{A}) \mid d \in \mathbb{Q}_{>0}\}$$

(with $f \geq g$ defined as $f = g$ or $f > g$), then it is called a *polarization*.

§2.3. Kirillov–Reshetikhin modules and the fusion construction

Consider the subalgebras of $\mathbb{Q}(q)$,

$$\mathcal{A}_{\mathbb{Z}} = \{f(q)/g(q) \mid f(q), g(q) \in \mathbb{Z}[q],\ g(0) = 1\}, \quad K_{\mathbb{Z}} = \mathcal{A}_{\mathbb{Z}}[q^{-1}].$$

Let $U_q'(\mathfrak{g})_{K_{\mathbb{Z}}}$ denote the $K_{\mathbb{Z}}$-subalgebra of $U_q'(\mathfrak{g})$ generated by e_i, f_i, q^h for all $i \in I$ and $h \in P^{\vee}$. The following is a combination of [KKM+92, Prop. 2.6.1] and [KKM+92, Prop. 2.6.2].

Proposition 2.1. *Let M be a finite-dimensional integrable $U_q'(\mathfrak{g})$-module. Suppose M has a prepolarization $(\ ,\ )$ and a $U_q'(\mathfrak{g})_{K_{\mathbb{Z}}}$-submodule $M_{K_{\mathbb{Z}}}$ such that $(M_{K_{\mathbb{Z}}}, M_{K_{\mathbb{Z}}}) \subseteq K_{\mathbb{Z}}$. Assume $M \cong \bigoplus_{k=1}^{m} V(\overline{\lambda}_k)$ as $U_q(\mathfrak{g}_0)$-modules, with $\overline{\lambda}_k \in P_0^+$ for all k, such that there exists $u_k \in (M_{K_{\mathbb{Z}}})_{\lambda_k}$ such that $(u_k, u_\ell) \in \delta_{k\ell} + q\mathcal{A}$ and $\|e_i u_k\|^2 \in q_i^{-2\langle h_i, \lambda_k \rangle - 2} q\mathcal{A}$ for all $i \in I_0$. Then $(\ ,\ )$ is a polarization and for*

$$L = \{v \in M \mid \|v\|^2 \in \mathcal{A}\}, \quad B = \{b \in (M_{K_{\mathbb{Z}}} \cap L)/(M_{K_{\mathbb{Z}}} \cap qL) \mid (b, b)_0 = 1\},$$

where $(\ ,\ )_0 \colon L/qL \to \mathbb{Q}$ is the bilinear form induced by $(\ ,\ )$ and the pair (L, B) is a crystal pseudobase of M.

For an indeterminate z, let M_z denote the $U_q'(\mathfrak{g})$-module $\mathbb{Q}(q)[z, z^{-1}] \otimes M$, where e_i and f_i act by $z^{\delta_{0i}} \otimes e_i$ and $z^{-\delta_{0i}} \otimes f_i$ is called the *affinization module* of M. For $a \in \mathbb{Q}(q)$, define the *evaluation module* $M_a = M_z/(z-a)M_z$. For $v \in M$, let v_a denote the corresponding element in M_a (i.e., the projection of $1 \otimes v$). Let $W(\varpi_r)$ denote the *fundamental module* from [Kas02].

[1]For $U_q(\mathfrak{g})$-modules M, N, a pairing $(\ ,\ ) \colon M \times N \to \mathbb{Q}(q)$ that satisfies (2.1) is often called admissible.

Proposition 2.2 ([Kas02, Prop. 9.3]). *Consider nonzero* $a, b \in \mathbb{Q}(q)$ *such that* $a/b \in \mathcal{A}$. *Then for any* $r \in I_0$, *there exists a unique nonzero* $U_q'(\mathfrak{g})$*-module homomorphism*

$$R_{a,b} \colon W(\varpi_r)_a \otimes W(\varpi_r)_b \to W(\varpi_r)_b \otimes W(\varpi_r)_a$$

that satisfies $R_{a,b}(u_a \otimes u_b) = u_b \otimes u_a$ *for some nonzero* $u \in W(\varpi_r)_{\varpi_r}$. *The map* $R_{a,b}$ *is called the* (normalized) R-matrix *and satisfies the Yang–Baxter equation.*

Denote

$$W(\varpi_r; a_1, a_2, \ldots, a_m) = W(\varpi_r)_{a_1} \otimes W(\varpi_r)_{a_2} \otimes \cdots \otimes W(\varpi_r)_{a_m}.$$

Let $\kappa = s_i$ if $\mathfrak{g}$ is of untwisted affine type and $\kappa = 1$ if $\mathfrak{g}$ is of twisted affine type. Since the R-matrix satisfies the Yang–Baxter equation, we can define the map

$$R_s \colon W(\varpi_r; q^{\kappa(s-1)}, q^{\kappa(s-3)}, \ldots, q^{\kappa(1-s)}) \to W(\varpi_r; q^{\kappa(1-s)}, \ldots, q^{\kappa(s-3)}, q^{\kappa(s-1)})$$

by applying the R-matrix on every pair of factors according to the long element of the symmetric group on s letters $(q^{\kappa(s-1)}, q^{\kappa(s-3)}, \ldots, q^{\kappa(1-s)})$. Let $W^{r,s}$ denote the image of R_s, which is a simple $U_q'(\mathfrak{g})$-module [Kas02], and we call $W^{r,s}$ a *Kirillov–Reshetikhin (KR) module.* From [CP95b, CP98], the module $W^{r,s}$ satisfies the Drinfel'd polynomial characterization of the usual definition of a KR module.

Lemma 2.3 ([KKM⁺92, Lem. 3.4.1]). *Let* M_j *and* N_j, *for* $j = 1, 2$, *be* $U_q'(\mathfrak{g})$*-modules such that there exists a pairing* $(\ ,\ )_j \colon M_j \times N_j \to \mathbb{Q}(q)$ *satisfying* (2.1). *Then there exists a pairing* $(\ ,\ ) \colon (M_1 \otimes M_2) \times (N_1 \otimes N_2) \to \mathbb{Q}(q)$ *defined by*

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1 (u_2, v_2)_2$$

for all $u_j \in M_j$ *and* $v_j \in N_j$ *with* $j = 1, 2$, *that satisfies* (2.1).

Remark 2.4. We note that there exists a $u \in W(\varpi_r)_{\varpi_r}$ such that $\|u\|^2 = 1$ since there exists $1 \otimes u \in (W(\varpi_r)_z)_{\varpi_r}$ such that $\|1 \otimes u\|^2 = 1$ by [Kas91, Nak04] and we have $\|z^k \otimes u\|^2 = \|1 \otimes u\|^2 = \|u\|^2$ by [Nak04, Lem. 4.7] as $W(\varpi_r) = W(\varpi_r)_1$.

Proposition 2.5 ([KKM⁺92, Prop. 3.4.3]). *Let* $u \in W(\varpi_r)_{\varpi_r}$ *be a vector such that* $\|u\|^2 = 1$.

(1) *The pairing* $(\ ,\ ) \colon W^{r,s} \times W^{r,s} \to \mathbb{Q}(q)$ *constructed using Lemma 2.3 and the prepolarization on* $W^{r,1}$ *(see [Kas02]) is a nondegenerate prepolarization on* $W^{r,s}$.
(2) $\left\| R_s(u_{q^{\kappa(s-1)}} \otimes u_{q^{\kappa(s-3)}} \otimes \cdots \otimes u_{q^{\kappa(1-s)}}) \right\|^2 = 1.$

Table 1. The nodes r such that we show $B^{r,s}$ exists.

$\mathfrak{g}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$E_6^{(2)}$
r	3, 5	2, 6	1	4	4

(3) $\big((W^{r,s})_{K_\mathbb{Z}}, (W^{r,s})_{K_\mathbb{Z}}\big) \subseteq K_\mathbb{Z}$, *where*

$$(W^{r,s})_{K_\mathbb{Z}} = R_s\left(\bigotimes_{k=0}^{s-1} U_q'(\mathfrak{g})_{K_\mathbb{Z}} u_{q^{\kappa(s-1-2k)}}\right) \cap \left(\bigotimes_{k=0}^{s-1} U_q'(\mathfrak{g})_{K_\mathbb{Z}} u_{q^{\kappa(s-1-2k)}}\right)$$

is a $U_q'(\mathfrak{g})_{K_\mathbb{Z}}$*-submodule of* $W^{r,s}$.

§3. Existence of KR crystals

This section is devoted to proving our main result.

Theorem 3.1. *Let* r *be such that* $W^{r,s}$ *is multiplicity-free as a* $U_q(\mathfrak{g}_0)$*-module for all* $s \in \mathbb{Z}_{>0}$*. Then* $W^{r,s}$ *admits a crystal pseudobase. Moreover, the KR crystal* $B^{r,s}$ *exists.*

We prove Theorem 3.1 case by case. When r is adjacent to 0 or in the orbit of 0 under a Dynkin diagram automorphism, Theorem 3.1 was shown in [KKM+92]. Theorem 3.1 was shown in nonexceptional affine types [Oka07, OS08]. Thus, it remains to show Theorem 3.1 for the values given in Table 1.

From Propositions 2.5 and 2.1, it is sufficient to show for the $U_q(\mathfrak{g}_0)$-module decomposition $W^{r,s} \cong \bigoplus_{k=1}^{M} V(\overline{\lambda}_k)$ (where $\overline{\lambda}_k \in P_0^+$), there exists $u_k \in \big((W^{r,s})_{K_\mathbb{Z}}\big)_{\lambda_k}$ such that

(i) $(u_k, u_\ell) \in \delta_{k\ell} + q\mathcal{A}$ and
(ii) $\|e_i u_k\|^2 \in q_i^{-2\langle h_i, \lambda_k\rangle - 2} q\mathcal{A}$.

The $U_q(\mathfrak{g}_0)$-module decomposition of $W^{r,s}$ is given in [Cha01].

We require the following facts. Since the decomposition is multiplicity-free, we have $(u_k, u_\ell) = 0$ for all $k \neq \ell$ since $\mathrm{wt}(u_k) \neq \mathrm{wt}(u_\ell)$. Note that

$$[m] \in q^{1-m}\mathcal{A}, \qquad \begin{bmatrix} m \\ k \end{bmatrix}_q \in q^{-k(m-k)}\mathcal{A}.$$

Let M be a $U_q'(\mathfrak{g})$-module. We will use this variant of equation (2.1):

$$(e_i^{(k)} v, w) = q_i^{k(k-\langle h_i,\mu\rangle)} (v, f_i^{(k)} w), \tag{3.1a}$$

$$(f_i^{(k)} v, w) = q_i^{k(k+\langle h_i,\mu\rangle)} (v, e_i^{(k)} w) \tag{3.1b}$$

for all $w \in M_\mu$. We also require

$$f_i^{(a)} e_i^{(b)} v = \sum_{k=0}^{\min(a,b)} \begin{bmatrix} a - b - \langle h_i, \mu \rangle \\ k \end{bmatrix}_{q_i} e_i^{(b-k)} f_i^{(a-k)} v \tag{3.2}$$

for any $v \in M_\mu$, which follows from applying the defining relation on $[e_i, f_i]$. By applying equations (3.1), (3.2) and the bilinearity of $(\ ,\ )$, we have for any $v \in M_\mu$,

$$\begin{aligned} \|e_i v\|^2 &= q_i^{1-\langle h_i, \mu \rangle} (v, f_i e_i v) \\ &= q_i^{1-\langle h_i, \mu \rangle} (v, e_i f_i v + [-\langle h_i, \mu \rangle]_{q_i} v) \\ &= q_i^{1-\langle h_i, \mu \rangle} \big((v, e_i f_i v) + [-\langle h_i, \mu \rangle]_{q_i} (v, v) \big) \\ &= q_i^{1-\langle h_i, \mu \rangle} \left(q_i^{-(1+\langle h_i, \mu \rangle)} \|f_i v\|^2 + [-\langle h_i, \mu \rangle]_{q_i} \|v\|^2 \right). \end{aligned}$$

Thus, we have

$$\|e_i v\|^2 = q_i^{-2\langle h_i, \mu \rangle} \|f_i v\|^2 + q_i^{1-\langle h_i, \mu \rangle} [-\langle h_i, \mu \rangle]_{q_i} \|v\|^2 . \tag{3.3}$$

For the remainder of the proof, we let $u \in W_{s\varpi_r}^{r,s}$ be such that $\|u\|^2 = 1$, where the existence of such follows from Lemma 2.3 and Remark 2.4. We have

$$\|f_i u\|^2 = q_i^{1+\delta_{ir} s} (u, e_i f_i u) = q_i^{1+\delta_{ir} s} (u, [\delta_{ir} s]_{q_i} u) = q_i^{1+\delta_{ir} s} [\delta_{ir} s]_{q_i} \tag{3.4}$$

for all $i \in I_0$ by equation (3.1a), the defining relation on $[e_i, f_i]$ (or equation (3.2)) and $e_i u = 0$. So we have $\|f_r u\|^2 \in q_r^2 \mathcal{A}$ (note $f_i u = 0$ for all $i \neq r$).

§3.1. Type $E_6^{(1)}$, $r = 3$

We claim that the elements

$$u_k := e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u$$

are the desired elements, where $0 \le k \le s$. We have

$$\mathrm{wt}(u_k) = \lambda_k := (s-k)\Lambda_3 + k\Lambda_6 - (2s-k)\Lambda_0,$$

and from [Cha01], the classical decomposition is $W^{3,s} \cong \bigoplus_{k=0}^{s} V\big((s-k)\overline{\Lambda}_3 + k\overline{\Lambda}_6\big)$. Thus, we need to show that u_k satisfies (i) and (ii).

We first show (i). We have

$$\|u_k\|^2 = q_6^{k(k-k)} (e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u, f_6^{(k)} u_k)$$

from equation (3.1a). Next we have

$$
\begin{aligned}
f_6^{(k)} u_k &= f_6^{(k)} e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_6} e_6^{(k-m)} f_6^{(k-m)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u \\
&= e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u,
\end{aligned} \tag{3.5}
$$

where the second equality comes from equation (3.2) and the third equality follows from the fact $e_i f_j = f_j e_i$ for all $i \neq j$ and $f_6 u = 0$ (so only the $m = k$ term is nonzero). By computations similar to equation (3.5) we have

$$
\|u_k\|^2 = (e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u, e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u) = \left\| e_0^{(k)} u \right\|^2 .
$$

Moreover, similar to equation (3.5), we have

$$
\begin{aligned}
\left\| e_0^{(k)} u \right\|^2 &= (e_0^{(k)} u, e_0^{(k)} u) = q_0^{k(k+2s-2k)} (u, f_0^{(k)} e_0^{(k)} u) \\
&= q_0^{k(2s-k)} \sum_{m=0}^{k} \begin{bmatrix} 2s \\ m \end{bmatrix}_{q_0} (u, e_0^{(k-m)} f_0^{(k-m)} u) = q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0} (u, u)
\end{aligned}
$$

since $f_0 u = 0$. Hence, we have

$$
\|u_k\|^2 = q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0} \in 1 + q\mathcal{A}. \tag{3.6}
$$

Next we show (ii). Fix some $i \in I_0$. From equation (3.3) it remains to compute $\|f_i u_k\|^2$. We compute $\|f_i u_k\|^2$ depending on the value of i. We note that the case of $k = 0$ is done by equation (3.4). Therefore, we assume $k \geq 1$. For $i = 6$ we have

$$
\begin{aligned}
f_6 u_k &= \begin{bmatrix} 1-k+k \\ 1 \end{bmatrix}_{q_6} e_6^{(k-1)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u + e_6^{(k)} f_6 e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u \\
&= e_6^{(k-1)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u
\end{aligned} \tag{3.7}
$$

by equation (3.2) and the fact $f_6 u = 0$. Hence, similar to the computation for $\|u_k\|^2$, we have

$$
\begin{aligned}
\|f_6 u_k\|^2 &= \left\| e_6^{(k-1)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u \right\|^2 \\
&= q_6^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_6} \left\| e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u \right\|^2 \\
&= q_6^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_6} q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0} .
\end{aligned}
$$

For $i = 1$ we have $f_1 u_k = e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} f_1 u = 0$, and so $\|f_1 u_k\|^2 = 0$. For $i = 5, 4, 2$ we have $f_i u_k = 0$ by applying equation (3.2) and the Serre relations (e.g., a straightforward calculation shows $e_4^{(k)} e_2^{(k-1)} e_0^{(k)} u = 0$ by repeatedly applying the Serre relations). Finally, we have $f_3 u_k = e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u$. Therefore, we have $\|f_3 u_k\|^2 = \left\|e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u\right\|^2$ similar to equation (3.5). However, for removing $e_4^{(k)}$, we obtain

$$(e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u, e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u) = q_4^{k(k-(k+1))} (e_2^{(k)} e_0^{(k)} f_3 u, f_4^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u)$$

by equation (3.1a). Furthermore, we have

$$\begin{aligned} f_4^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u &= \sum_{m=0}^{k} \begin{bmatrix} k-1 \\ m \end{bmatrix}_{q_4} e_4^{(k-m)} f_4^{(k-m)} e_2^{(k)} e_0^{(k)} f_3 u \\ &= \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_{q_4} e_4 e_2^{(k)} e_0^{(k)} f_4 f_3 u + \begin{bmatrix} k-1 \\ k \end{bmatrix}_{q_4} e_2^{(k)} e_0^{(k)} f_3 u \\ &= e_4 e_2^{(k)} e_0^{(k)} f_4 f_3 u, \end{aligned}$$

where we note that $\begin{bmatrix} k-1 \\ k \end{bmatrix}_{q_4} = 0$ (recall that we assumed $k \geq 1$). Thus, by applying equation (3.1a) we obtain

$$\begin{aligned} \left\| e_4^{(k)} e_2^{(k)} e_0^{(k)} f_3 u \right\|^2 &= q_4^{-k} (e_2^{(k)} e_0^{(k)} f_3 u, e_4 e_2^{(k)} e_0^{(k)} f_4 f_3 u) \\ &= q_4^{-k} q_4^{k} \left\| e_2^{(k)} e_0^{(k)} f_4 f_3 u \right\|^2. \end{aligned} \tag{3.8}$$

Next we have

$$\left\| e_2^{(k)} e_0^{(k)} f_4 f_3 u \right\|^2 = \left\| e_0^{(k)} f_2 f_4 f_3 u \right\|^2$$

from a similar computation to equation (3.8). Continuing using equation (3.1a), we have

$$\left\| e_0^{(k)} f_2 f_4 f_3 u \right\|^2 = q_0^{k(2s-1-k)} (f_2 f_4 f_3 u, f_0^{(k)} e_0^{(k)} f_2 f_4 f_3 u).$$

We note that $f_0 f_2 f_4 f_3 w = 0$ for any $w \in W_{\varpi_3}^{3,1}$ from weight considerations (the resulting element would have classical weight $\overline{\Lambda}_1 + \overline{\Lambda}_5$, which is not in $\overline{\Lambda}_3 + Q^-$ by [Kas02, Thm. 5.17]) and the classical decomposition. So $f_0 f_2 f_4 f_3 (w_1 \otimes \cdots \otimes w_s) = 0$ for any $w_1, \ldots, w_s \in W_{\varpi_3}^{3,1}$ from applying the coproduct $\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$. Thus, we have $f_0 f_2 f_4 f_3 u = 0$ from the construction of u and $W^{3,s}$. Therefore, we compute

$$f_0^{(k)} e_0^{(k)} f_2 f_4 f_3 u = \sum_{m=0}^{k} \begin{bmatrix} 2s-1 \\ m \end{bmatrix}_{q_0} e_0^{(k-m)} f_0^{(k-m)} f_2 f_4 f_3 u = \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} f_2 f_4 f_3 u$$

similar to equation (3.5) and using the Serre relations. Thus, we have

$$\left\| e_0^{(k)} f_2 f_4 f_3 u \right\|^2 = q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \| f_2 f_4 f_3 u \|^2 .$$

Next we see

$$\begin{aligned} \| f_2 f_4 f_3 u \|^2 &= q_2^{1-1} (f_4 f_3 u, e_2 f_2 f_4 f_3 u) = (f_4 f_3 u, [1]_{q_2} f_4 f_3 u) \\ &= q_4^{1-1} (f_3 u, e_4 f_4 f_3 u) = (f_3 u, [1]_{q_4} f_3 u) = \| f_3 u \|^2 \end{aligned}$$

by a similar computation to equation (3.4). Hence, we have

$$\begin{aligned} \| f_3 u_k \|^2 &= q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \| f_3 u \|^2 \\ &= q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} q_3^{1+s} [s]_{q_3} \in q_3^2 \mathcal{A}, \end{aligned} \tag{3.9}$$

where the last equality is by equation (3.4). To complete the proof of (ii) we can see that

$$\begin{gathered} q_i^{-2\langle h_i, \lambda_k \rangle} \| f_i u_k \|^2 \in q_i^{-2\langle h_i, \lambda_k \rangle} \mathcal{A}, \\ q_i^{1-\langle h_i, \lambda_k \rangle} [-\langle h_i, \lambda_k \rangle]_{q_i} q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \in q_i^2 \mathcal{A}, \end{gathered}$$

noting $\langle h_i, \lambda_k \rangle \geq 0$.

§3.2. Type $E_6^{(1)}$, $r = 5$

The following are the desired elements in $W^{5,s}$:

$$u_k := e_1^{(k)} e_3^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u_0 \in W^{5,s}_{(s-k)\varpi_5 + k\varpi_1},$$

where $0 \leq k \leq s$. The proof is the same as $r = 3$ after applying the order 2 diagram automorphism that fixes 0.

§3.3. Type $E_7^{(1)}$, $r = 2$

The following are the desired elements in $W^{2,s}$:

$$u_k := e_7^{(k)} e_6^{(k)} e_5^{(k)} e_4^{(k)} e_3^{(k)} e_1^{(k)} e_0^{(k)} u_0 \in W^{2,s}_{(s-k)\varpi_2 + k\varpi_7},$$

where $0 \leq k \leq s$. The proof is similar to $W^{3,s}$ in type $E_6^{(1)}$, where we compute

$$\| u_k \|^2 = q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0},$$

$$\|f_7 u_k\|^2 = q_7^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_7} \|u_k\|^2,$$
$$\|f_i u_k\|^2 = 0 \quad (i = 6, 5, 4, 3, 1),$$
$$\|f_2 u_k\|^2 = q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \|f_2 u\|^2.$$

§3.4. Type $E_6^{(2)}$, $r = 4$

We claim

$$u_{k',k} := e_0^{(k')} e_1^{(k)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u$$

are the desired elements, where $0 \leq k' \leq k \leq s$. We note that

$$\mathrm{wt}(u_{k',k}) = \lambda_{k',k} := (s-k)\Lambda_4 + (k-k')\Lambda_1 - (2s-2k')\Lambda_0.$$

To obtain the parameterization of the classical decomposition

$$W^{4,s} \cong \bigoplus_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 \leq s}} V(t_1 \overline{\Lambda}_4 + t_2 \overline{\Lambda}_1)$$

given in [Scr20, Prop. 9.31], we set $t_1 = s - k$ and $t_2 = k - k'$ (which is forced by weight considerations). Note that $t_1 \geq 0$ if and only if $k \leq s$; $t_2 \geq 0$ if and only if $k' \leq k$; and $t_1 + t_2 \leq s$ if and only if $0 \leq k'$ (as $t_1 + t_2 = s - k'$). Hence, we have the same classical decomposition.

To show (i) we have

$$\|u_{0,k}\|^2 = q_1^{k(k-k)} (e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u, f_1^{(k)} u_{0,k}).$$

Next we compute

$$\begin{aligned}
f_1^{(k)} u_{0,k} &= f_1^{(k)} e_1^{(k)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_1} e_1^{(k-m)} f_1^{(k-m)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_1} e_1^{(k-m)} e_2^{(k)} e_3^{(k)} e_2^{(k)} \sum_{p=0}^{k-m} \begin{bmatrix} k-m \\ p \end{bmatrix}_{q_1} e_1^{(k-p)} f_1^{(k-m-p)} e_0^{(k)} u \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_1} \begin{bmatrix} k-m \\ k-m \end{bmatrix}_{q_1} e_1^{(k-m)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(m)} e_0^{(k)} u \\
&= e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u,
\end{aligned}$$

where the last equality follows from the fact $e_2^{(k)} e_1^{(m)} e_0^{(k)} u = 0$ for all $k > m$ by the Serre relations and $e_2 u = 0$. Hence, we have

$$\begin{aligned}\|u_{0,k}\|^2 &= \left\| e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u \right\|^2 \\ &= q^{k(k-k)} (e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u, f_2^{(k)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u).\end{aligned}$$

Now, similar to the previous computation for $u' = e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u$, we obtain

$$\begin{aligned} f_2^{(k)} e_2^{(k)} u' &= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_2} e_2^{(k-m)} f_2^{(k-m)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u \\ &= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_2} e_2^{(k-m)} e_3^{(k)} \sum_{p=0}^{k-m} \begin{bmatrix} k-m \\ p \end{bmatrix}_{q_2} e_2^{(k-p)} f_2^{(k-m-p)} e_1^{(k)} e_0^{(k)} u \\ &= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q_2} \begin{bmatrix} k-m \\ k-m \end{bmatrix}_{q_2} e_2^{(k-m)} e_3^{(k)} e_2^{(m)} e_1^{(k)} e_0^{(k)} u \\ &= e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u = u' \end{aligned}$$

since $e_3^{(k)} e_2^{(m)} e_1^{(k)} e_0^{(k)} u = 0$ for all $k > m$ by the Serre relations (recall that $A_{32} = -1$) and $e_3 u = 0$. Hence, we have

$$\|u_{0,k}\|^2 = \left\| e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u \right\|^2 = q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0} \in 1 + q\mathcal{A},$$

where the last equality is shown similarly to equation (3.6).

Next we consider

$$\|u_{k',k}\|^2 = q_0^{k'(k'+2s-2k')} (u_{0,k}, f_0^{(k')} u_{k',k}).$$

We compute

$$f_0^{(k')} u_{k',k} = f_0^{(k')} e_0^{(k')} u_{0,k} = \sum_{m=0}^{k'} \begin{bmatrix} 2s \\ m \end{bmatrix}_{q_0} e_0^{(k'-m)} f_0^{(k'-m)} u_{0,k}, \tag{3.10}$$

and

$$\begin{aligned} f_0^{(k'-m)} e_0^{(k)} u &= \sum_{p=0}^{k'-m} \begin{bmatrix} k'-m-k+2s \\ p \end{bmatrix}_{q_0} e_0^{(k-p)} f_0^{(k'-m-p)} u \\ &= \begin{bmatrix} k'-m-k+2s \\ k'-m \end{bmatrix}_{q_0} e_0^{(k-k'+m)} u \end{aligned}$$

as $k' - m \le k$ (since $k' \le k$ and $m \ge 0$) and $f_0 u = 0$. Next we have $e_1^{(k)} e_0^{(m)} u = 0$ for all $k > m$ by the Serre relations and $e_1 u = 0$, and so the only term that is nonzero in equation (3.10) is when $m = k'$. Therefore, we have

$$\|u_{k',k}\|^2 = q_0^{k'(2s-k')} \begin{bmatrix} 2s \\ k' \end{bmatrix}_{q_0} \|u_{0,k}\|^2 = q_0^{k'(2s-k')} \begin{bmatrix} 2s \\ k' \end{bmatrix}_{q_0} q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0} \in 1 + q\mathcal{A}.$$

To show (ii) it remains to compute $\|f_i u_{k',k}\|^2$ by equation (3.3), and by equation (3.4), we can assume $k \geq 1$. For $i \in I_0$ we have $f_i u_{k',k} = e_0^{(k')} f_i u_{0,k}$, and by the above we have

$$\|f_i u_{k',k}\|^2 = q_0^{k'(2s-\delta_{i1}-k')} \begin{bmatrix} 2s-\delta_{i1} \\ k' \end{bmatrix}_{q_0} \|f_i u_{0,k}\|^2.$$

Next, similar to the computation in equation (3.7), we have

$$\begin{aligned} f_1 u_{0,k} &= e_1^{(k-1)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u + e_1^{(k)} e_2^{(k)} e_3^{(k)} e_2^{(k)} f_1 e_1^{(k)} e_0^{(k)} u \\ &= e_1^{(k-1)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u + e_1^{(k)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k-1)} e_0^{(k)} u \\ &= e_1^{(k-1)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u, \end{aligned}$$

where the last equality uses $e_2^{(k)} e_1^{(m)} e_0^{(k)} u = 0$ for all $k > m$. Therefore, we have

$$\|f_1 u_{0,k}\|^2 = q_1^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_1} q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0}$$

by a computation similar to equation (3.6). Similar to equation (3.9) we have

$$\|f_4 u_{0,k}\|^2 = q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \|f_4 u\|^2.$$

We also have $f_2 u_{0,k} = f_3 u_{0,k} = 0$ by applying the Serre relations. Thus, we see that (ii) holds.

§3.5. Type $E_7^{(1)}$, $r = 6$

The following are the desired elements in $W^{6,s}$:

$$u_{k',k} := e_0^{(k')} e_1^{(k)} e_3^{(k)} e_4^{(k)} e_5^{(k)} e_2^{(k)} e_4^{(k)} e_3^{(k)} e_1^{(k)} e_0^{(k)} u \in W^{6,s}_{(s-t_1-t_2)\varpi_6 + t_2\varpi_1},$$

where $0 \leq k' \leq k \leq s$. Then $\mathrm{wt}(u_{k',k}) = (s-k)\Lambda_6 + (k-k')\Lambda_1 - (2s-2k')\Lambda_0$. Showing that the classical decomposition is the same as in [Cha01] is similar to the $r = 4$ case for type $E_6^{(2)}$. Moreover, it is similar to show that

$$\begin{aligned} \|u_{k',k}\|^2 &= q_0^{k'(2s-k')} \begin{bmatrix} 2s \\ k' \end{bmatrix}_{q_0} q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0}, \\ \|f_i u_{k',k}\|^2 &= q_0^{k'(2s-\delta_{i1}-k')} \begin{bmatrix} 2s-\delta_{i1} \\ k' \end{bmatrix}_{q_0} \|f_i u_{0,k}\|^2 \quad (i \in I_0), \end{aligned}$$

$$\begin{aligned}
\|f_1 u_{0,k}\|^2 &= q_1^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_1} \|u_{0,k}\|^2, \\
\|f_i u_{0,k}\|^2 &= 0 \quad (i = 2,3,4,5,7), \\
\|f_6 u_{0,k}\|^2 &= q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \|f_6 u\|^2.
\end{aligned}$$

§3.6. Type $E_8^{(1)}$, $r = 1$

The following are the desired elements in $W^{1,s}$:

$$u_{k',k} := e_0^{(k')} e_8^{(k)} e_7^{(k)} e_6^{(k)} e_5^{(k)} e_4^{(k)} e_3^{(k)} e_2^{(k)} e_4^{(k)} e_5^{(k)} e_6^{(k)} e_7^{(k)} e_8^{(k)} e_0^{(k)} u,$$

where $0 \le k' \le k \le s$. We take $u_{k',k} \in W^{1,s}_{(s-t_1-t_2)\varpi_1 + t_2\varpi_8}$. Then $\mathrm{wt}(u_{k',k}) = (s-k)\Lambda_1 + (k-k')\Lambda_8 - (2s-2k')\Lambda_0$. Showing that the classical decomposition is the same as in [Cha01] is similar to the $r = 4$ case for type $E_6^{(2)}$. Moreover, it is similar to show that

$$\begin{aligned}
\|u_{k',k}\|^2 &= q_0^{k'(2s-k')} \begin{bmatrix} 2s \\ k' \end{bmatrix}_{q_0} q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0}, \\
\|f_i u_{k',k}\|^2 &= q_0^{k'(2s-\delta_{i8}-k')} \begin{bmatrix} 2s-\delta_{i8} \\ k' \end{bmatrix}_{q_0} \|f_i u_{0,k}\|^2 \quad (i \in I_0), \\
\|f_8 u_{0,k}\|^2 &= q_8^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_8} \|u_{0,k}\|^2, \\
\|f_i u_{0,k}\|^2 &= 0 \quad (i = 2,3,4,5,6,7), \\
\|f_1 u_{0,k}\|^2 &= q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \|f_1 u\|^2.
\end{aligned}$$

§3.7. Type $F_4^{(1)}$, $r = 4$

The following are the desired elements in $W^{4,s}$:

$$u_{k',k} := e_0^{(k')} e_1^{(k)} e_2^{(k)} e_3^{(2k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u \in W^{4,s}_{(s-2k)\varpi_4 + (k-k')\varpi_1},$$

where $0 \le k' \le k \le s/2$. Then $\mathrm{wt}(u_{k',k}) = (s-2k)\Lambda_4 + (k-k')\Lambda_1 - (s-2k')\Lambda_0$. To obtain the parameterization of the classical decomposition

$$W^{4,s} \cong \bigoplus_{t_2=0}^{s/2} \bigoplus_{t_1=0}^{t_2} V\big((s-2t_2)\overline{\Lambda}_4 + t_1\overline{\Lambda}_1\big)$$

given in [Cha01], we take $t_1 = k - k'$ and $t_2 = k$. Indeed, we have $t_2 \le s/2$ if and only if $k \le s/2$; $t_1 \ge 0$ if and only if $k \le k'$; and $t_1 \le t_2$ if and only if $0 \le k'$.

Moreover, it is similar to the $r = 4$ case for type $E_6^{(2)}$ to show that

$$\begin{aligned}
\|u_{k',k}\|^2 &= q_0^{k'(2s-k')} \begin{bmatrix} 2s \\ k' \end{bmatrix}_{q_0} q_0^{k(2s-k)} \begin{bmatrix} 2s \\ k \end{bmatrix}_{q_0}, \\
\|f_i u_{k',k}\|^2 &= q_0^{k'(2s-\delta_{i1}-k')} \begin{bmatrix} 2s-\delta_{i1} \\ k' \end{bmatrix}_{q_0} \|f_i u_{0,k}\|^2 \quad (i \in I_0), \\
\|f_1 u_{0,k}\|^2 &= q_1^{k-1} \begin{bmatrix} k \\ k-1 \end{bmatrix}_{q_1} \|u_{0,k}\|^2, \\
\|f_2 u_{0,k}\|^2 &= \|f_3 u_{0,k}\|^2 = 0, \\
\|f_4 u_{0,k}\|^2 &= q_0^{k(2s-1-k)} \begin{bmatrix} 2s-1 \\ k \end{bmatrix}_{q_0} \|f_4 u\|^2.
\end{aligned}$$

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